

Topologically Massive Abelian Gauge Theory

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Abstract:

We discuss three mathematical structures which arise in topologically massive abelian gauge theory. First, the euclidean topologically massive abelian gauge theory defines a contact structure on a manifold. We briefly discuss three solutions and the related contact structures on the flat 3-torus, the AdS space, the 3-sphere which respectively correspond to Bianchi type *I*, *VIII*, *IX* spaces. We also present solutions on Bianchi type *II*, *VI* and *VII* spaces. Secondly, we discuss a family of complex (anti-)self-dual solutions of the euclidean theory in cartesian coordinates on \mathcal{R}^3 which are given by (anti-)holomorphic functions. The orthogonality relation of contact structures which are determined by the real parts of these complex solutions separates them into two classes: the self-dual and the anti-self-dual solutions. Thirdly, we apply the curl transformation to this theory. An arbitrary solution is given by a vector tangent to a sphere whose radius is determined by the topological mass in transform space. Meanwhile a gauge transformation corresponds to a vector normal to this sphere. We discuss the quantization of topological mass on an example.

1 Introduction

Topologically massive gauge theories are dynamical theories which are specific to three dimensions [1, 2], [3]. They are qualitatively different from the Yang-Mills type gauge theories beside their mathematical elegance and consistency. They provide an alternative way to introduce mass term with no spontaneous symmetry breaking. The most distinctive feature of the topologically massive gauge theories is the existence of a natural scale of length which is determined by the inverse of the topological mass [4, 5], [6, 7, 8]. This leads to purely geometric discussions of these theories [7, 8].

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We discuss three mathematical structures in this context. The effect of the topological mass is to introduce a natural scale of length into these structures. To the knowledge of the author, the connection of these structures with the topologically massive gauge theories has been overlooked in the literature. Our results further contribute to discussions of these structures.

First we discuss the connection of the topologically massive abelian gauge theory with contact geometry. A real-valued topologically massive abelian gauge potential on a Riemannian (euclidean signature) manifold is a Beltrami (Trkalian) field [9] which defines a contact structure. We discuss three solutions and the relevant contact structures arising on the flat 3-torus \mathcal{T}^3 , the AdS space (lorentzian) \mathcal{H}^3 , the 3-sphere \mathcal{S}^3 which respectively correspond to Bianchi type *I*, *VIII*, *IX* spaces. We also present solutions on Bianchi type *II*, *VI* and *VII* spaces. We realize the Darboux contact form as a topologically massive gauge potential in euclidean (type *II* and *VII*) and lorentzian (type *VI*) signatures. The Bianchi type *II*, *VI*, *VII*, *VIII* and *IX* spaces are homogeneous contact manifolds [10], [11, 12]. We further present complex-valued or lorentzian solutions which do not lead to contact structures on Bianchi type *V* and specialized forms of Bianchi type *VI* and *VII* spaces.

Secondly, we discuss a family of complex Beltrami (Trkalian) fields including the topological mass in cartesian coordinates on \mathcal{R}^3 [13]. The complex (anti-)self-dual solutions of the euclidean (and also lorentzian) theory are given by (anti-)holomorphic functions. The orthogonality relation of contact structures which are determined by the real parts of these complex solutions separates them into two classes: the self-dual and the anti-self-dual solutions. These two classes, in real case, possess opposite helicities.

Thirdly, we apply the curl transformation [14], [15, 16] to this theory. This is based on decomposing a solution into (complex) helical eigen-functions of the curl operator in the fashion of a Fourier transform refining the Helmholtz's decomposition. An arbitrary field is given by a vector tangent to a sphere whose radius is determined by the topological mass in transform space [15], [17]. Meanwhile a gauge transformation corresponds to a vector normal to this sphere in this space. We also discuss the sense of quantization of topological mass on an example.

The references [17, 18] and [19, 20] contain nice, elementary expositions of some of the relevant basic mathematical concepts in different contexts.

2 Contact Structures and Beltrami Fields

A contact form α on a three dimensional manifold \mathcal{M} is a 1-form such that $\alpha \wedge d\alpha$ vanishes nowhere: $\alpha \wedge d\alpha \neq 0$. A contact structure ξ is a smooth tangent plane field on \mathcal{M} which is locally given as the kernel of a contact 1-form: $\xi = \ker(\alpha)$. Thus a manifold \mathcal{M} endowed with a globally defined contact structure is called a contact manifold [21], [22, 23], [24, 25, 26]. A tangent plane in ξ at a point of \mathcal{M} is a contact element. It follows from the Frobenius integrability theorem that a contact structure is a maximally non-integrable tangent plane field on \mathcal{M} . The unique vector field \vec{X} which satisfies $\alpha(\vec{X}) = 1$ and $\mathcal{L}_{\vec{X}}\alpha = \vec{X} \lrcorner d\alpha = 0$ is called the characteristic or Reeb vector field associated with the contact structure. Further if the Reeb vector is a Killing vector: $\mathcal{L}_{\vec{X}}g = 0$ for a Riemannian metric g on \mathcal{M} then this is called a K-contact structure [22].

A diffeomorphism f on a contact manifold \mathcal{M} is a contact transformation (contactomorphism) if it preserves the contact structure. More precisely, two contact structures $\xi_\alpha = \ker(\alpha)$ and $\xi_\beta = \ker(\beta)$ on a manifold \mathcal{M} are contactomorphic if there exists a contact transformation f so that $f^*\beta = \mu\alpha$ for some non-vanishing function μ on \mathcal{M} or equivalently $f_*\xi_\alpha = \xi_\beta$ [23].

Darboux theorem asserts that there exists local coordinates about any point on a contact manifold \mathcal{M} in which the contact form can be expressed as $\alpha = dz + xdy$ [21]. In other words all contact structures are locally contactomorphic. Thus they locally look alike in Darboux coordinates [27]. However these are implicitly global objects [28].

The eigen-vectors of the curl operator

$$\vec{\nabla} \times \vec{\alpha} = \lambda \vec{\alpha}, \quad (1)$$

on \mathcal{R}^3 are called Beltrami vectors [9]. The eigen-value λ is a function on \mathcal{R}^3 . If this is a constant then the eigen-vector is called Trkalian [13]. We refer the reader to [29] for a derivation from a variational principle of this equation for constant eigen-value. We can associate a 1-form $\alpha = \alpha_i dx^i$ with the vector $\vec{\alpha} = \alpha^i \partial_i$ if we *adapt* the metric $g_{ij} = \text{diag}(1, 1, 1)$ [30]. The metric induces the usual correspondence between vectors and 1-forms. Then we can write the equation (1) using differential forms as

$$*d\alpha - \lambda\alpha = 0, \quad (2)$$

($\vec{\nabla} \times \leftrightarrow *d$). If α is Trkalian then $d * \alpha = 0$, ($\vec{\nabla} \cdot \leftrightarrow d*$). Thus a Beltrami 1-form α defines a contact structure since $\alpha \wedge d\alpha = \lambda\alpha \wedge *\alpha \neq 0$ [27]. This expression defines a volume form on \mathcal{R}^3 . The contact structure is positively (negatively) oriented if this orientation agrees (disagrees) with the orientation on \mathcal{R}^3 , depending on the sign of λ .

The helicity of the 1-form α is defined as

$$H(\alpha) = \int_D \alpha \wedge d\alpha, \quad (3)$$

where D is a domain in \mathcal{R}^3 [9]. The helicity density $\mathcal{H}(\alpha) = \alpha \wedge d\alpha$ is a local measure of *twisting* of the smooth contact structure defined by α so as to be maximally non-integrable [17]. The contact field has positive (negative) helicity if $\lambda > 0$ ($\lambda < 0$). The helicity (3) corresponds to an abelian Chern-Simons term in gauge theory [31, 32].

It has been frequently observed in the literature that a curl eigen-vector is dual to a contact form [28, 33, 34, 35] (and the references therein), (see also [17, 18] and [19, 20]). A more precise correspondence follows by adapting a metric to a contact form as introduced in [30]. This is also related to contact metric structures [22]. A Riemannian metric is said to be adapted to the contact form α if α is of unit *length* and it satisfies the equation (2) for $\lambda = 2$ [30]. We shall use a slightly more general definition by unrestricting the length of α and λ .

A homogeneous manifold with an invariant contact structure is called a homogeneous contact manifold. More precisely, if there exist a metric associated with the contact 1-form α and a group of diffeomorphisms acting transitively as a group of isometries which leave α invariant then we have a homogeneous contact manifold [10]. We refer the reader to [30] for the existence of adapted metrics on any Riemannian contact manifold.

The self-duality equation in three dimensions

$$(*d - \nu)A = *F - \nu A = 0, \quad (4)$$

[36, 37], is the generalization of the eigen-form equation (2) of the $*d$ operator to a curved manifold \mathcal{M} with an adapted metric $g_{\mu\nu}$ and a gauge potential $A = \alpha$ on it, for a constant eigen-value $\nu = \lambda$. The field equation of the topologically massive abelian gauge theory

$$(*d - \nu) * F = *d(*d - \nu)A = 0, \quad (5)$$

is given by applying the operator $*d$ on the self-duality equation (4), [7]. Thus a real-valued self-dual solution of the topologically massive abelian gauge theory on a Riemannian (euclidean signature) manifold \mathcal{M} is given by a Beltrami (Trkalian) gauge potential. Accordingly this gauge potential defines a contact structure on \mathcal{M} . We remark that $\alpha \wedge *\alpha$ is positive definite on a Riemannian manifold \mathcal{M} if α is real-valued. But this is not true if α is complex-valued or \mathcal{M} is of lorentzian signature. We shall call: $(*d + \nu)A = *F + \nu A = 0$ the anti-self-duality equation in anticipation with (4). The examples below consist of solutions on Bianchi type spaces. These are homogeneous spaces.

Furthermore, the contact structure defined by the 1-form $*F$ in (4) coincides with that of A . After all, a contact structure is defined up to a multiple (ν) of the contact 1-form $\alpha = A$ ($\beta = \nu A = *dA = *F$). Note that the field equation (5) is simply the self-duality equation written for the contact 1-form $*F$ since the equations (4) and (5) are symmetric under the interchange $\nu A \leftrightarrow *F$, [7]. The helicity densities of the potential A and the dual-field $*F$ are related as $\mathcal{H}(*F) = \nu^2 \mathcal{H}(A)$. The helicity density is a gauge-dependent quantity. The self-dual model which is separately introduced in [36] is related to the topologically massive abelian theory by a Legendre transformation [37].

The contact structure defined by the gauge potential A (or dual-field $*F$) is locally contactomorphic to the contact structure defined by the gauge Darboux form $A' = f^*A = dz + xdy$. In other words there exists local coordinates in which the topologically massive contact gauge potential 1-form is given by A' . Then the self-duality equation in these coordinates

$$(*_{g'}d - \nu)A' = 0, \quad (6)$$

is also satisfied adapting the pull-back metric $g' = f^*g$ since $f^* *_g dA = *_{g'} f^* dA = *_{g'} df^* A$. Here f is an orientation preserving contactomorphism.

2.1 The 3-sphere

A self-dual solution of the euclidean topologically massive abelian gauge theory on the 3-sphere \mathcal{S}^3 is given in [6], [7]. The 3-sphere which is locally given as $\mathcal{S}^1 \times \mathcal{S}^2$ is a Bianchi type IX space [6]. The effect of the topological mass is to introduce a natural scale of length $r = 2/\nu$ where $\nu = ng^2$ [7]. The contact gauge potential is given by

$$A = -\frac{1}{2} \frac{\nu}{g} \omega^3 = -\frac{1}{2} \frac{\nu}{g} [d\psi + \cos(\nu\theta)d\phi]. \quad (7)$$

Here

$$\begin{aligned} \omega^1 &= -\sin(\nu\psi)d\theta + \cos(\nu\psi)\sin(\nu\theta)d\phi, & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= \cos(\nu\psi)d\theta + \sin(\nu\psi)\sin(\nu\theta)d\phi, \\ \omega^3 &= d\psi + \cos(\nu\theta)d\phi, \end{aligned} \quad (8)$$

are the *modified* left-invariant basis 1-forms of $SU(2)$ which is parameterized in terms of the Eulerian (half) arclengths θ , ϕ and ψ corresponding to the Euler angles $\tilde{\theta} = \nu\theta$, $\tilde{\phi} = \nu\phi$ and $\tilde{\psi} = \nu\psi$ on the 3-sphere \mathcal{S}^3 of radius r [7]. The metric is given by

$$ds^2 = \eta_{ab}\omega^a\omega^b. \quad (9)$$

The Maurer-Cartan equation: $d\omega^a = -\frac{1}{2}C_b^a\omega^b \wedge \omega^c$ for the basis 1-forms yields the non-vanishing structure constants: $C_2^1{}_3 = -\nu$, $C_3^2{}_1 = -\nu$, $C_1^3{}_2 = -\nu$ of the Bianchi type IX group which are modified by the topological mass ν . The Maurer-Cartan equation and Hodge duality relations for the basis 1-forms immediately lead to the result that $\alpha = A$ satisfies the self-duality equation.

The 1-form $\alpha = A$ defines the standard contact structure on \mathcal{S}^3 [23], [24, 25, 26, 35]. The contact plane field $\xi = \ker(A) = \text{Span}\{\vec{e}_1, \vec{e}_2\}$ is orthogonal to the trajectory of the vector \vec{e}_3 which is a Hopf fiber [28]. Here $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the frame dual to the co-frame $\{\omega^1, \omega^2, \omega^3\}$. Further the Reeb vector is given by \vec{e}_3 [27], which is also a Killing vector for the metric. Thus the Hopf contact gauge potential A defines a K-contact structure on

\mathcal{S}^3 . Note that ω^1 and ω^2 are also self-dual Beltrami (Trkalian) 1-forms [6]. These also define contact structures on \mathcal{S}^3 [38].

The scaling of the unmodified basis 1-forms by the radius of the 3-sphere is also noted in [39], [40].

2.2 The Flat 3-Torus

The next example of a topologically massive [Beltrami (Trkalian)] gauge potential is the *abc-flow* on the flat 3-torus \mathcal{T}^3 [9], [27]. Consider the *abc-potential*

$$A = \frac{\nu}{g} (U dx + V dy + W dz), \quad (10)$$

$$\begin{aligned} U &= a \sin(\nu z) + c \cos(\nu y), \\ V &= b \sin(\nu x) + a \cos(\nu z), \\ W &= c \sin(\nu y) + b \cos(\nu x), \end{aligned}$$

with the topological mass ν on the flat 3-torus \mathcal{T}^3 . Here a , b and c are arbitrary constants. The flat 3-torus

$$\mathcal{T}^3 = \{(x, y, z) \in \mathcal{R}^3; \text{mod } 2\pi/\nu\}, \quad (11)$$

is given by identifying the opposite faces of a cube of edge length $2\pi/\nu$ in \mathcal{R}^3 . This can be embedded in a five-dimensional sphere \mathcal{S}^5 in \mathcal{R}^6 [22]. The metric inherited from the natural flat metric on \mathcal{R}^6 coincides with the flat metric on \mathcal{R}^3 in which the cube resides. Thus this (more precisely the covering space \mathcal{R}^3) is a Bianchi type *I* space

$$\begin{aligned} \omega^1 &= dx, & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= dy, \\ \omega^3 &= dz. \end{aligned} \quad (12)$$

The effect of the topological mass is again to introduce a natural scale of length $r = 1/\nu$ where $\nu = ng^2$. The parameters x , y and z are the arclengths

corresponding to the angles $\tilde{x} = \nu x$, $\tilde{y} = \nu y$, $\tilde{z} = \nu z$ on \mathcal{T}^3 . This potential is a self-dual solution (4) of the euclidean topologically massive abelian gauge theory (5) on \mathcal{T}^3 . Thus it defines a contact structure.

A special contact structure on \mathcal{T}^3 is given by $b = c = 0$, $a = 1$ and a re-labelling: $x \leftrightarrow y$ in (10)

$$A = \cos(\nu z)dx + \sin(\nu z)dy, \quad (13)$$

(ignoring ν/g) [41]. As defined on $\mathcal{S}^1 \times \mathcal{R}^2$ this (13), for $\nu = 1$, is associated with the space of contact elements in the plane \mathcal{R}^2 which is isomorphic to the spherized cotangent bundle of the plane [42], [43, 44], [45], [46].

The cases $c = a = 0$ and $a = b = 0$ similarly define contact structures. The scaling of the unmodified abc-flow is also noted in [47].

2.2.1 Bianchi Type *II*, *VI* and *VII* Spaces

We can also realize this (13) as a topologically massive gauge potential on a space with the modified left-invariant basis 1-forms

$$\begin{aligned} \omega^1 &= \cos(\nu z)dx + \sin(\nu z)dy, & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= \sin(\nu z)dx - \cos(\nu z)dy, \\ \omega^3 &= dz, \end{aligned} \quad (14)$$

of Bianchi type *VII*. This yields the Euclidean metric (9). The Maurer-Cartan equation for the basis 1-forms yields the structure constants: $C_2^1{}_3 = -\nu$, $C_3^2{}_1 = -\nu$ of the Bianchi type *VII* group which are modified by the topological mass. This reduces to the group of isometries of the Euclidean plane for $\nu = 1$. The 1-forms $\alpha = A = \omega^1$ and $\beta = \omega^2$ satisfy the self-duality equation.

Furthermore, this contact structure is related to the standard contact structure defined by the Darboux contact 1-form $\alpha = dz + xdy$ via

$$(x, y, z) \longrightarrow \left(z \cos(\nu y) + \frac{1}{\nu} x \sin(\nu y), z \sin(\nu y) - \frac{1}{\nu} x \cos(\nu y), y \right), \quad (15)$$

[26]. The pull-back basis 1-forms are

$$\begin{aligned}
\omega^1 &= dz + xdy, \\
\omega^2 &= \frac{1}{\nu}dx - \nu zdy, \\
\omega^3 &= dy,
\end{aligned} \tag{16}$$

and the metric (9). The euclidean, self-dual topologically massive Darboux gauge potential is given by $A = \alpha = \omega^1$.

The lorentzian version of (14) is given by the modified left-invariant basis 1-forms

$$\begin{aligned}
\omega^0 &= \cosh(\nu z)dx + \sinh(\nu z)dy, & \eta_{ab} &= \text{diag}(-1, 1, 1), \\
\omega^1 &= \sinh(\nu z)dx + \cosh(\nu z)dy, \\
\omega^2 &= dz,
\end{aligned} \tag{17}$$

of Bianchi type VI which yields the Minkowski metric (9). The Maurer-Cartan equation yields the structure constants: $C_{12}^0 = \nu$, $C_{20}^1 = -\nu$. This reduces to the group of isometries of the Minkowski plane for $\nu = 1$. In this case both $\alpha = A = \omega^0$ and $\beta = \omega^1$ satisfy the anti-self-duality equation.

The contact transformation to the Darboux form $\alpha = dz + xdy$ is given by

$$(x, y, z) \longrightarrow \left(z \cosh(\nu y) + \frac{1}{\nu} x \sinh(\nu y), -z \sinh(\nu y) - \frac{1}{\nu} x \cosh(\nu y), y \right). \tag{18}$$

This yields the basis 1-forms

$$\begin{aligned}
\omega^0 &= dz + xdy, \\
\omega^1 &= -\frac{1}{\nu}dx - \nu zdy, \\
\omega^2 &= dy,
\end{aligned} \tag{19}$$

with the metric (9). In this case the lorentzian, anti-self-dual topologically massive Darboux gauge potential is given by $A = \alpha = \omega^0$.

We can also realize the Darboux contact form $\alpha = A = dz + xdy$ as an euclidean, self-dual topologically massive gauge potential on a space with the modified left-invariant basis 1-forms

$$\begin{aligned}\omega^1 &= dz + xdy, & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= \frac{1}{\nu}dx, \\ \omega^3 &= dy,\end{aligned}\tag{20}$$

of Bianchi type *II* and the metric (9). The Maurer-Cartan equation yields the structure constant: $C_2^1{}_3 = -\nu$. The Bianchi type *II* group, $\nu = 1$, is also known as the Heisenberg group.

A hyperbolic version of the abc-potential (10) is possible only if $b = 0$ with a twist of sign in c :

$$A = [a \sinh(\nu z) + c \cosh(\nu y)] dx + a \cosh(\nu z) dy - c \sinh(\nu y) dz, \tag{21}$$

on the Minkowski space (17). The 1-form A is anti-self-dual. The case $c = 0$, $a = 1$ can be realized on Bianchi type *VI* space (17) as in the euclidean case.

2.2.2 Bianchi Type *V* and Specialized Type *VI*, *VII* Spaces

As we have already remarked, a real-valued solution on a lorentzian manifold or a complex-valued solution on an euclidean manifold do not necessarily lead to contact structures. In the three examples below the solution 1-forms α yield $\alpha \wedge d\alpha = 0$.

If we adapt a specialized form

$$\begin{aligned}\omega^0 &= e^{\pm \frac{\nu}{2} z} \left[\cosh\left(\frac{\nu}{2} z\right) dx + \sinh\left(\frac{\nu}{2} z\right) dy \right], & \eta_{ab} &= \text{diag}(-1, 1, 1), \\ \omega^1 &= e^{\pm \frac{\nu}{2} z} \left[\sinh\left(\frac{\nu}{2} z\right) dx + \cosh\left(\frac{\nu}{2} z\right) dy \right], \\ \omega^2 &= dz,\end{aligned}\tag{22}$$

of the modified left-invariant basis 1-forms of Bianchi type *VI* (17) [48] with the metric (9) then $\alpha = A = \omega^0 \pm \omega^1$ is an anti-self-dual solution. The Maurer-Cartan equation yields the structure constants: $C_1^0{}_2 = \nu/2$, $C_2^1{}_0 = -\nu/2$,

$C_{02}^0 = \pm\nu/2$, $C_{12}^1 = \pm\nu/2$. An anisotropic version of the frame (22) with $\nu = 2$ yields a solution of the topologically massive gravity [48].

For the euclidean version of (22) with the specialized form

$$\begin{aligned}\omega^1 &= e^{\pm\frac{\nu}{2}z} \left[\cos\left(\frac{\nu}{2}z\right) dx + \sin\left(\frac{\nu}{2}z\right) dy \right], & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= e^{\pm\frac{\nu}{2}z} \left[\sin\left(\frac{\nu}{2}z\right) dx - \cos\left(\frac{\nu}{2}z\right) dy \right], \\ \omega^3 &= dz,\end{aligned}\tag{23}$$

of the modified left-invariant basis 1-forms of Bianchi type *VII* (14) and the metric (9), the complex 1-form $\alpha = A = \omega^1 \pm i\omega^2$ is a self-dual solution with a complex-valued topological mass: $\nu \longrightarrow (1 - i)\nu/2$ in (4). The Maurer-Cartan equation yields the structure constants: $C_{23}^1 = -\nu/2$, $C_{31}^2 = -\nu/2$, $C_{13}^1 = \pm\nu/2$, $C_{23}^2 = \pm\nu/2$.

If we adapt the modified left-invariant basis 1-forms

$$\begin{aligned}\omega^1 &= e^{-\nu z} dx, & \eta_{ab} &= \text{diag}(1, 1, 1), \\ \omega^2 &= e^{-\nu z} dy, \\ \omega^3 &= dz,\end{aligned}\tag{24}$$

of Bianchi type *V* with the metric (9), then $\alpha = A = \omega^0 \pm i\omega^1$ is a (anti-)self-dual solution with a complex-valued topological mass: $\nu \longrightarrow \pm i\nu$ in (4). The Maurer-Cartan equation yields the structure constants: $C_{13}^1 = -\nu$, $C_{23}^2 = -\nu$.

2.3 The Anti-de Sitter Space

An anti-self-dual, lorentzian solution of the topologically massive abelian gauge theory on AdS space \mathcal{H}^3 is discussed in [6], [8]. The AdS space is a Bianchi type *VIII* space [6]. It is globally given as $\mathcal{S}^1 \times \mathcal{H}_+^2$ where \mathcal{H}_+^2 is the upper portion of a hyperboloid of two sheets in \mathcal{R}^3 . The contact gauge potential is given by

$$A = -\frac{1}{2} \frac{\nu}{g} \omega^3 = -\frac{1}{2} \frac{\nu}{g} [d\psi + \cosh(\nu\theta)d\phi].\tag{25}$$

Here

$$\begin{aligned}
\omega^1 &= -\cos(\nu\psi)d\theta - \sin(\nu\psi)\sinh(\nu\theta)d\phi, & \eta_{ab} &= \text{diag}(-1, -1, 1), \\
\omega^2 &= -\sin(\nu\psi)d\theta + \cos(\nu\psi)\sinh(\nu\theta)d\phi, & & \\
\omega^3 &= d\psi + \cosh(\nu\theta)d\phi, & &
\end{aligned} \tag{26}$$

are the modified left-invariant basis 1-forms of $SU(1, 1)$ which is parameterized in terms of the *Eulerian* (half) arclengths θ , ϕ and ψ corresponding to the Euler parameters $\tilde{\theta} = \nu\theta$, $\tilde{\phi} = \nu\phi$ and $\tilde{\psi} = \nu\psi$ on the AdS space \mathcal{H}^3 of radius $r = 2/\nu$, $\nu = ng^2$ [8]. The Maurer-Cartan equation for these basis 1-forms yields the structure constants: $C_{2\,3}^1 = -\nu$, $C_{3\,1}^2 = -\nu$, $C_{1\,2}^3 = \nu$ of the Bianchi type *VIII* group which are modified by the topological mass. This is the lorentzian analogue [49] of the euclidean solution (7) on \mathcal{S}^3 [8]. The gauge potential $\alpha = A$ (25) defines an analogous [50] lorentzian K-contact structure on \mathcal{H}^3 [51, 52, 53]. The contact planes are time-like [52] with the conventions of [8]. Note that ω^1 and ω^2 are also anti-self-dual Beltrami (Trkalian) 1-forms.

We can also realize this using another set of modified invariant basis 1-forms

$$\begin{aligned}
\omega^1 &= \frac{1}{\nu} \frac{1}{y} [\cos(\nu z)dx + \sin(\nu z)dy], & \eta_{ab} &= \text{diag}(-1, -1, 1), \\
\omega^2 &= \frac{1}{\nu} \frac{1}{y} [\sin(\nu z)dx - \cos(\nu z)dy], & & \\
\omega^3 &= \frac{1}{\nu} \frac{1}{y} (dx + \nu y dz), & &
\end{aligned} \tag{27}$$

of Bianchi type *VIII* [54]. The Maurer-Cartan equation for these basis 1-forms yields the same set of structure constants. The 1-forms $\alpha = \omega^3$ and $\beta = \omega^1$, $\gamma = \omega^2$ are anti-self-dual.

The Bianchi type *II*, *VI*, *VII*, *VIII* and *IX* spaces above with their respective invariant contact structures determined by $\alpha = A$ and the metrics adapted so as to satisfy the (anti-)self-duality equation are among the basic examples of homogeneous contact manifolds [10].

3 Duality and Holomorphic Functions on \mathcal{R}^3

In this section, we adapt the classification of the complex Trkalian fields ($\nu = 1$) which is based on Clebsch decomposition of a 1-form in terms of Monge potentials [13]. See the appendix for a brief explanation. We refer the reader to [13] and [55], [56] for a detailed discussion of the Clebsch decomposition. We shall focus on the cartesian case including the topological mass. In cartesian coordinates a real (anti-)self-dual 1-form α can be written as the real part of the complex (anti-)self-dual 1-form

$$a = e^{i\nu z} df, \quad (28)$$

$\alpha = \text{Re}\{a\}$. Here z is the third component of the cartesian coordinates and $f : R^2 \longrightarrow C$ is a complex-valued function on the xy -plane. Then the (anti-)self-duality equation

$$\begin{aligned} *da - \nu a = 0 : f_{w*} = 0, \quad w = x + iy, \\ (*da + \nu a = 0 : f_w = 0), \end{aligned} \quad (29)$$

yields that $f = u + iv$ is a (anti-)holomorphic function. Thus complex (anti-)self-dual solutions (28) of the euclidean topologically massive abelian gauge theory on \mathcal{R}^3 in cartesian coordinates are given by (anti-)holomorphic functions. Their real parts accordingly yield real contact structures. Note that we have a freedom of permutation of the x , y and the z coordinates for writing such solutions. The solution (28) has planar fronts along the z -axis as equiphase surfaces. In the Minkowski space with signature $(+, +, -)$ the (anti-)holomorphicity conditions of f for the solution (28) of the (anti-)self-duality equation (29) interchange.

For example the anti-holomorphic function $f = x - iy$ yields the contact 1-form (13)

$$A = \text{Re}\{e^{i\nu z} df\}, \quad f = x - iy. \quad (30)$$

Meanwhile for the holomorphic function $f = x + iy$ we have

$$\begin{aligned}
A &= \text{Re}\{e^{i\nu z}df\}, & f &= x + iy, \\
&= \cos(\nu z)dx - \sin(\nu z)dy.
\end{aligned} \tag{31}$$

Note that a change in the z -direction: $a = \exp(-i\nu z)df$ interchanges the holomorphicity conditions since this corresponds to a complex conjugation in the xy -plane.

The contact structure defined by a real, self-dual 1-form α is positively oriented: $\alpha \wedge d\alpha > 0$ (assume $\nu > 0$) while a contact structure defined by an anti-self-dual, real 1-form is negatively oriented: $\alpha \wedge d\alpha < 0$. These are related by a change of orientation. Further, a real, self-dual or anti-self-dual 1-form α respectively represents a contact structure with positive or negative helicity. The helicity density of the complex solution (28) vanishes: $\mathcal{H}(a) = 0$. The helicity densities $\mathcal{H}(\alpha)$ and $\mathcal{H}(\beta)$ of its real and complex parts: $\alpha = \text{Re}\{a\}$, $\beta = \text{Co}\{a\}$ are given as

$$\mathcal{H}(\alpha) = \mathcal{H}(\beta) = \begin{cases} \nu(u_x^2 + u_y^2) * 1 & \text{self-dual} \\ -\nu(u_x^2 + u_y^2) * 1 & \text{anti-self-dual} \end{cases} \tag{32}$$

where $*1 = dx \wedge dy \wedge dz$.

3.1 Orthogonal Contact Structures and Duality

We can construct real orthogonal contact structures as follows. Consider the complex 1-forms $a = e^{i\nu z}df$ and $b = e^{-i\nu z}dg$. There exist orthogonal contact structures in case $f = u + iv$ is holomorphic and $g = p + iq$ is anti-holomorphic or vice-versa. That is a and b are both self-dual or anti-self-dual. In both cases the orthogonality condition reduces to a single equation: $\vec{\nabla}u \cdot \vec{\nabla}p = u_x p_x + u_y p_y = 0$. We can use the simplest solution: $b = ia^*$ of this equation for constructing the orthogonal contact structures. Their real parts

$$\begin{aligned}
\alpha &= \text{Re}\{a\} = \cos(\nu z)du - \sin(\nu z)dv \\
&= [u_x \cos(\nu z) - v_x \sin(\nu z)] dx + [u_y \cos(\nu z) - v_y \sin(\nu z)] dy,
\end{aligned}$$

(33)

$$\begin{aligned}\beta &= \text{Re}\{b\} = \sin(\nu z)du + \cos(\nu z)dv \\ &= [v_x \cos(\nu z) + u_x \sin(\nu z)] dx + [v_y \cos(\nu z) + u_y \sin(\nu z)] dy,\end{aligned}$$

yield the orthogonal contact structures

$$\xi_\alpha = \text{Span}\{\vec{U}, \vec{S}\}, \quad \xi_\beta = \text{Span}\{\vec{V}, \vec{S}\}, \quad (34)$$

$$\begin{aligned}\vec{U} &= (u_y \cos(\nu z) - v_y \sin(\nu z), -u_x \cos(\nu z) + v_x \sin(\nu z), 0), \\ \vec{V} &= (v_y \cos(\nu z) + u_y \sin(\nu z), -v_x \cos(\nu z) - u_x \sin(\nu z), 0), \\ \vec{S} &= (0, 0, 1).\end{aligned}$$

The 1-forms α and β are related to du and dv in uv -plane by a $U(1)$ rotation

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos(\nu z) & -\sin(\nu z) \\ \sin(\nu z) & \cos(\nu z) \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}. \quad (35)$$

However there exists no orthogonal structures arising from the distinct duality classes. Thus the orthogonality relation of the contact structures determined by the real parts of the set of complex 1-form solutions $a = e^{\mp i\nu z} df$ separates this set into self-dual and anti-self-dual classes. Therefore we call these contact structures (anti-)self-dual. Note that a rotation of ξ_β about the vector \vec{V} produces another set of planes which are also orthogonal to ξ_α . But its intersection with the solution set only contains ξ_β .

A simple choice for the orthogonality condition is: $p_x = \mp u_x$, $p_y = \pm u_y$. This leads to the equations

$$(u_x)^2 - (u_y)^2 = 0, \quad (p_x)^2 - (p_y)^2 = 0. \quad (36)$$

Then a consistency check yields the solutions

$$\begin{aligned}a_1 &= e^{i\nu z} d[x + y - i(x - y)], & b_1 &= e^{-i\nu z} d[x - y - i(x + y)], \\ a_2 &= e^{i\nu z} d[x + y + i(x - y)], & b_2 &= e^{-i\nu z} d[x - y + i(x + y)],\end{aligned} \quad (37)$$

which are respectively self-dual and anti-self-dual. Their complex conjugates are also solutions.

4 The Curl Transformation

In this section, we apply the curl transformation [14] which is readily developed in the context of force-free magnetic fields or Beltrami (Trkalian) fields [15, 16] to the euclidean topologically massive abelian gauge theory in the vector form. We also discuss quantization of the topological mass: $\nu = ng^2$ on an example.

The curl transformation is based on decomposing a vector field into (complex) helical eigen-functions

$$\vec{\chi}_\lambda(\vec{x}|\vec{k}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \vec{Q}_\lambda(\vec{k}), \quad (38)$$

of the curl operator (1) with planar fronts along a direction determined by the vector $\vec{k} = (k_1, k_2, k_3)$ [57]. The vectors $\vec{Q}_\lambda(\vec{k})$ are given as

$$\begin{aligned} \vec{Q}_\lambda(\vec{k}) &= -\frac{\lambda}{\sqrt{2}} \left(\frac{k_1(k_1 + i\lambda k_2)}{k(k + k_3)} - 1, \frac{k_2(k_1 + i\lambda k_2)}{k(k + k_3)} - i\lambda, \frac{k_1 + i\lambda k_2}{k} \right), \quad \lambda = \pm 1, \\ \vec{Q}_0(\vec{k}) &= -\frac{\vec{k}}{k}, \quad k = |\vec{k}|. \end{aligned} \quad (39)$$

The parameter λ corresponds to sense of helicity states. These form a complex basis in Fourier space [14]. The basis vectors $\vec{Q}_\lambda(\vec{k})$ are undefined when $k + k_3 = 0$. The eigen-functions $\vec{\chi}_\lambda(\vec{x}|\vec{k})$ of the curl operator

$$\begin{aligned} \vec{\nabla} \times \vec{\chi}_\lambda(\vec{x}|\vec{k}) &= k\lambda \vec{\chi}_\lambda(\vec{x}|\vec{k}), \\ \vec{\nabla} \cdot \vec{\chi}_\lambda(\vec{x}|\vec{k}) &= 0, \quad \lambda = \pm 1, \\ \vec{\nabla} \cdot \vec{\chi}_0(\vec{x}|\vec{k}) &= -\frac{1}{(2\pi)^{3/2}} i k e^{i\vec{k}\cdot\vec{x}}, \end{aligned} \quad (40)$$

form an orthogonal and complete set [14]. See the appendix for a brief account. Any vector field can be represented in terms of these eigen-functions in the fashion of a (vector) Fourier transform refining the Helmholtz decomposition [14]. According to Helmholtz theorem we can decompose a vector field

into divergence-free $[\vec{\chi}_\lambda(\vec{x}|\vec{k}), \lambda = \pm 1]$ and curl-free $[\vec{\chi}_0(\vec{x}|\vec{k})]$ components. The divergence-free component further consists of two helicity states: positive ($\lambda = +1$) and negative ($\lambda = -1$). The curl transformation is based on a helicity decomposition in the basis determined by $\vec{Q}_\lambda(\vec{k})$ [17]. We refer the reader to [14] for the motivation of introduction of this basis. We shall use the divergence-free components for expressing the field and the potential and the curl-free component for the gauge transformation.

4.1 The Gauge Field

An euclidean topologically massive field \vec{F}

$$\vec{\nabla} \times \vec{F} - \nu \vec{F} = 0, \quad (41)$$

can be expressed as

$$\vec{F}(\vec{x}) = \sum_{\lambda} ' \vec{F}_{\lambda}(\vec{x}), \quad (42)$$

where

$$\vec{F}_{\lambda}(\vec{x}) = \frac{1}{g} \int \vec{\chi}_{\lambda}(\vec{x}|\vec{k}) f_{\lambda}(\vec{k}) d^3 k, \quad (43)$$

excluding the divergenceful component [15, 16]. The factor $1/g$ leads to the correct strength for the gauge potential as we shall discuss on an example. In this decomposition each helicity component is given in terms of a scalar function

$$f_{\lambda}(\vec{k}) = g \int \vec{\chi}_{\lambda}^*(\vec{x}|\vec{k}) \cdot \vec{F}(\vec{x}) d^3 x, \quad (44)$$

[14]. If we replace the expression (42) in the equation (41) we find

$$f_{\lambda}(\vec{k}) = \frac{\delta(k - \lambda\nu)}{k^2} s_{\lambda}(\vec{k}), \quad (45)$$

using a radial delta function [15]. Thus an arbitrary solution is given entirely in terms of its transform on the sphere of radius $k = \lambda\nu = |\nu|$. Furthermore, only the eigen-functions for which $\lambda = \text{sgn}(\nu)$ contribute to the field (42), [15]. Then the expansion (42) or (43) simplifies into

$$\begin{aligned}\vec{F}_\lambda(\vec{x}) &= \frac{1}{g} \int \vec{\chi}_\lambda(\vec{x}|\lambda\nu\vec{\kappa}) s_\lambda(\lambda\nu\vec{\kappa}) d\Omega \\ &= \frac{1}{(2\pi)^{3/2}} \frac{1}{g} \int e^{i\lambda\nu\vec{\kappa}\cdot\vec{x}} \vec{Q}_\lambda(\vec{\kappa}) s_\lambda(\lambda\nu\vec{\kappa}) d\Omega,\end{aligned}\tag{46}$$

where $d\Omega$ is the spherical area element and $\vec{\kappa} = \frac{\vec{k}}{k}$ is a unit vector in transform space. Therefore a solution can be defined entirely by the value of its curl transform on the unit sphere in transform space [15]. We call s_λ the spherical curl transform in order to distinguish it from the full curl transform f_λ [16].

A simple example [15] is given by $s_\lambda(\lambda\nu\vec{\kappa}) = s_0\lambda h(\nu)\delta(\vec{\kappa} - \vec{\kappa}_0)$ where $s_0 = \sqrt{2}(2\pi)^{3/2}$ and $\vec{\kappa}_0 = (0, 0, 1)$. Here $h(\nu)$ is a function which will be determined in accordance with the strength of the gauge potential \vec{A}_λ . This yields

$$\vec{F}_\lambda(\vec{x}) = \frac{1}{g} h e^{i\lambda\nu z} (1, i\lambda, 0).\tag{47}$$

In order to exhibit the distributional character of both sides in (44) if we multiply by e^{ikp} and integrate over k [replacing (45)], we find

$$s_\lambda(\lambda\nu\vec{\kappa}) = \frac{1}{(2\pi)^{1/2}} g\nu^2 e^{-i\lambda\nu p} \vec{Q}_\lambda^*(\vec{\kappa}) \cdot \vec{F}_\lambda^R(p, \vec{\kappa}),\tag{48}$$

where

$$\vec{F}_\lambda^R(p, \vec{\kappa}) = \int \vec{F}_\lambda(\vec{x}) \delta(p - \vec{\kappa} \cdot \vec{x}) d^3x,\tag{49}$$

is the (vector) Radon transform of $\vec{F}_\lambda(\vec{x})$. This basically is the integral of $\vec{F}_\lambda(\vec{x})$ over the planes at a distance $p = \vec{\kappa} \cdot \vec{x}$ to the origin with unit

normal $\vec{\kappa}$. Thus for a fixed $\vec{\kappa}$ this corresponds to a *plane wave* that is a function constant on planes orthogonal to $\vec{\kappa}$ [58], [59]. The description of the spherical curl transform [and its inverse (46)] in terms of the Radon transform (48) is both necessary and sufficient [16].

We can easily check the spherical curl transform of the self-dual solutions $\vec{a}_1 = (1 - i)e^{i\nu z}(1, i)$ (37) and $\vec{c}_1 = \vec{b}_1^* = i\vec{a}_1$. Their transforms are respectively $s_{+1}^a(\nu\vec{\kappa}) = s_0(1 - i)\delta(\vec{\kappa} - \vec{\kappa}_0)$ and $s_{+1}^c(\nu\vec{\kappa}) = is_{+1}^a(\nu\vec{\kappa})$ (factor of g ignored) where $s_0 = \sqrt{2}/(2\pi)^{3/2}$ and $\delta(\vec{\kappa} - \vec{\kappa}_0)$ is a delta function acting at the point $\vec{\kappa}_0 = \vec{e}_3$ on the unit sphere in transform space. Meanwhile the transform of $\vec{b}_1 = -i\vec{a}_1^*$ is given as $s_{-1}^b(-\nu\vec{\kappa}) = i[s_{+1}^a(\nu\vec{\kappa})]^*$.

In the anti-self-dual case of the equation (41) which contains an extra factor of $(-)$, we include this factor also in the equations (45), (46) and (48). In this case only the eigen-functions with opposite helicity that is for which $\lambda = -\text{sgn}(\nu)$ contribute to the field. A simple example is given by $s_\lambda(-\lambda\nu\vec{\kappa}) = s_0(-\lambda)h(\nu)\delta(\vec{\kappa} - \vec{\kappa}_0)$ where $s_0 = -\sqrt{2}/(2\pi)^{3/2}$ and $\vec{\kappa}_0 = (0, 0, 1)$. This yields

$$\vec{F}_\lambda(\vec{x}) = \frac{1}{g} h e^{-i\lambda\nu z} (1, i\lambda, 0). \quad (50)$$

4.2 The Gauge Potential

The gauge potential for the field (43) is given by

$$\vec{A}_\lambda(\vec{x}) = \frac{1}{g} \lambda \int \vec{\chi}_\lambda(\vec{x}|\vec{k}) f_\lambda(\vec{k}) \frac{1}{k} d^3k, \quad (51)$$

[14]. If we replace the equation (45) in this, we find the gauge potential

$$\vec{A}_\lambda(\vec{x}) = \frac{1}{g} \frac{1}{\nu} \int \vec{\chi}_\lambda(\vec{x}|\lambda\nu\vec{\kappa}) s_\lambda(\lambda\nu\vec{\kappa}) d\Omega, \quad (52)$$

for the field $\vec{F}_\lambda(\vec{x})$ (46). This satisfies the self-duality equation $\vec{F}_\lambda = \nu \vec{A}_\lambda$ where $\vec{F}_\lambda = \vec{\nabla} \times \vec{A}_\lambda$. We find

$$\lambda f_\lambda(\vec{k}) \frac{1}{k} = g \int \vec{\chi}_\lambda^*(\vec{x}|\vec{k}) \cdot \vec{A}_\lambda(\vec{x}) d^3x, \quad (53)$$

inverting the equation (51). If we multiply this by e^{ikp} and integrate over k [replacing (45)], we find

$$s_\lambda(\lambda\nu\vec{\kappa}) = \frac{1}{(2\pi)^{1/2}} g\nu^3 e^{-i\lambda\nu p} \vec{Q}_\lambda^*(\vec{\kappa}) \cdot \vec{A}_\lambda^R(p, \vec{\kappa}), \quad (54)$$

which can also be inferred from the self-duality equation and (48).

The potential for the example (47) is

$$\vec{A}_\lambda(\vec{x}) = \frac{1}{g} \frac{h}{\nu} e^{i\lambda\nu z} (1, i\lambda, 0). \quad (55)$$

This corresponds to the complex solution (31) in differential forms for $\lambda = +1$ (ignoring the strength $h/g\nu$). Meanwhile for $\lambda = -1$ with an extra $(-)$ sign [see equation (50)] in ν this corresponds to (30).

Note that the analysis from the equations (41) to (49) can be directly applied to the self-duality equation for the potential. For the abc-potential/dual-field (10) the vector $\vec{\kappa}_0$ consists of three components along the x , y and z -axis [15].

4.3 The Gauge Transformation

We need a curl-free vector $\vec{\nabla}U$ for a gauge transformation

$$\vec{A}' = \vec{A} - \frac{1}{g} \vec{\nabla}U, \quad (56)$$

of the potential. A gauge transformation should be consistent not only with the field equation (41) but its integral $\vec{F}' - \nu\vec{A}' = (1/g)\nu\vec{\nabla}U$ (self-duality equation with a source-like term) as well [7, 8]. Thus we have

$$\vec{\nabla}U(\vec{x}) = \frac{1}{\nu} \int \vec{\chi}_0(\vec{x}|\vec{k}) f_0(\vec{k}) d^3k. \quad (57)$$

If we ignore the factor $1/\nu$ here, then the dimensions of $f_\lambda(\vec{k})$, $\lambda = \pm 1$ in (43), (51) and $f_0(\vec{k})$ which would be included in the expression for a general

vector field become inconsistent. Further, this is necessary for the equations (52) and (56) to be consistent. This yields the gauge function

$$U(\vec{x}) = \frac{1}{(2\pi)^{3/2}} i \frac{1}{\nu} \int e^{i\vec{k} \cdot \vec{x}} f_0(\vec{k}) \frac{1}{k} d^3k. \quad (58)$$

In connection with the examples (47) and (55), we can require the gauge function to take values in \mathcal{S}^1 with radius $r = 1/\nu$ of $\mathcal{S}^1 \times \mathcal{R}^2$ which is mentioned in sections 2.2 and 3. Then $f_0(\vec{k}) = f_0(k\vec{\kappa}) = f_0 h(k) \frac{\delta(k-\lambda\nu)}{k^2} \delta(\vec{\kappa} - \vec{\kappa}_0)$ where $f_0 = (2\pi)^{3/2}$ and $\vec{\kappa}_0 = (0, 0, 1)$. This yields

$$\vec{\nabla} U_\lambda(\vec{x}) = -\frac{h}{\nu} e^{i\lambda\nu z} (0, 0, 1), \quad (59)$$

where

$$U_\lambda(\vec{x}) = i\lambda \frac{h}{\nu^2} e^{i\lambda\nu z}. \quad (60)$$

The (curl transform of) gauge transformation (59) acts at the same point as $\vec{A}_\lambda(\vec{x})$, (55) $[\vec{F}_\lambda(\vec{x})$, (47)] on the transform sphere with this choice of $f_0(\vec{k})$. The dimensions of $s_\lambda(\lambda\nu\vec{\kappa})$ for the examples (47), (55) and $f_0(\vec{k})$ for (59) are consistent as required by the equation (45). The introduction of the parameter λ in U is just for convenience in the examples.

The field (46), the potential (52) and the gauge transformation (57) are respectively expressed in terms of the tangential and the normal vectors on the sphere in transform space. We can see this from the equations $\vec{\nabla} \cdot \vec{\chi}_\lambda(\vec{x}|\vec{k}) = i\vec{k} \cdot \vec{\chi}_\lambda(\vec{x}|\vec{k}) = 0$, $\lambda = \pm 1$ [$\vec{k} \cdot \vec{Q}_\lambda(\vec{k}) = 0$] and $\vec{\nabla} \times \vec{\chi}_0(\vec{x}|\vec{k}) = i\vec{k} \times \vec{\chi}_0(\vec{x}|\vec{k}) = 0$, $\lambda = 0$ [$\vec{k} \times \vec{Q}_0(\vec{k}) = 0$]. Thus we can think of the (vector) Fourier transform of the field and the potential as tangent to the sphere in the transform space [17], [60] whereas the gauge transformation corresponds to a normal vector. The gauge potential \vec{A}' (56) consists of both components. Note that, choice of \vec{k} in (57) determines the form of the gauge transformation $\vec{\nabla} U(\vec{x})$. One could choose, as an example, another \vec{k} in (57) different from that in (46) and (52) since the gauge transformation is just an arbitrary gradient vector. Then, this would again yield a normal vector. The precise form of a gauge transformation for a specific example depends on its geometric features.

4.4 Discussion of the Example

We can discuss the sense of quantization of the topological mass and the strength of the gauge potential on the examples (47), (55) and (59) following the reasoning in [7]. If we choose $h(\nu) = \nu^2$ [$h(k) = k^2$] then the gauge potential (55) and its gauge transform

$$\vec{A}'_\lambda = \frac{\nu}{g} e^{i\lambda\nu z} (1, i\lambda, 1), \quad (61)$$

have the same strength ν/g . Thus the strength of the gauge potential will be given by the gauge coupling constant $\nu/g = ng$ if $\nu = ng^2$ [7, 8]. This leads us to adopt a fundamental scale of length $R = 1/g^2$ (radius) or $L = 2\pi/g^2$ (perimeter) [7]. We can write the relation $\nu = ng^2$ as $\nu = 2\pi n/L$. If the gauge potential and the gauge transformation that is the factor

$$e^{i\lambda\nu z} = e^{i\lambda\frac{2\pi n}{L}z}, \quad (62)$$

is a single-valued function of z with the fundamental scale L then n has to be an integer. The fundamental length scale L is the least common multiple of intervals over which the gauge potential and the transformation are single-valued and periodic for any integer n in addition to the fact that they have a smaller period $l = L/n$. We can associate the integer n with the winding number of a map $G : \mathcal{S}_L^1 \longrightarrow \mathcal{S}_l^1$ which winds the circle \mathcal{S}_L^1 of perimeter L about the circle \mathcal{S}_l^1 of perimeter l with locally invariant arclength [7]. We remark that we can use a similar reasoning for the abc-potential (10). This has already been discussed in the context of fluid dynamics. See for example [47, 61].

4.5 A Source-like Term

Furthermore, if we are given a solution \vec{F} of the field equation (41) which is expressed as in (46), then we can introduce a source-like term

$$\vec{\nabla} \times \vec{F}' - \nu \vec{F}' = \vec{J}, \quad (63)$$

choosing $\vec{F}' = \vec{F} - (1/g)\vec{\nabla}V$ and $\vec{J} = (\nu/g)\vec{\nabla}V$ with a reasoning similar to a gauge transformation. Then we find $\vec{\nabla} \cdot \vec{J} = -\nu\vec{\nabla} \cdot \vec{F}' = (\nu/g)\nabla^2 V$. We can similarly express this term as

$$\vec{J}(\vec{x}) = \frac{\nu}{g} \int \vec{\chi}_0(\vec{x}|\vec{k}) f_0(\vec{k}) d^3 k. \quad (64)$$

Thus a divergenceful $[\vec{\chi}_0(\vec{x}|\vec{k})]$ term in (42) which was previously excluded corresponds to a source-like term in the field equation (41). This term yields a vector normal to the sphere in the transform space. If V is a harmonic function then: $\vec{\nabla} \cdot \vec{J} = -\nu\vec{\nabla} \cdot \vec{F}' = 0$. If we choose $V = 1/x$ where $x = |\vec{x}|$, this yields: $\vec{\nabla} \cdot \vec{F}' = (4\pi/g)\delta(\vec{x})$.

5 Conclusion

We have discussed three structures in topologically massive abelian gauge theory. The most distinctive feature of the topologically massive gauge theories is the existence of a natural scale of length which is determined by the inverse topological mass. The abelian gauge theory reduces to the study of Beltrami (Trkalian) fields on a manifold with an adapted metric once this is recognized.

Thus the topologically massive abelian gauge theory on a Riemannian manifold defines a contact structure. This is locally contactomorphic to the standard contact structure defined by the Darboux form. Therefore a topologically massive (anti-)self-dual gauge potential or dual-field on a Riemannian manifold is locally given by the Darboux form with an adapted metric. In other words this theory on a Riemannian manifold locally has a unique solution up to contactomorphisms. These contact structures locally look alike in Darboux coordinates. In this sense, the topologically massive abelian gauge theory is implicitly the study of various local (gauge theoretic, physical etc.) aspects with an adapted metric of these contactomorphisms. Nevertheless these structures can possess different global features.

We have presented solutions on Bianchi type *I*, *II*, *V*, *VI*, *VII*, *VIII* and *IX* spaces. We have briefly described the contact structures defined by the gauge potential on the flat 3-torus (Bianchi *I*), the AdS space (Bianchi *VIII*) and the 3-sphere (Bianchi *IX*). The Bianchi type *II*, *VI*, *VII*, *VIII*, and

IX spaces with their respective invariant contact structures and the metrics adapted so as to satisfy the (anti-)self-duality equation are the examples of homogeneous contact manifolds. We have realized the Darboux contact form as a topologically massive gauge potential in euclidean (type II and VII) and lorentzian (type VI) signatures. We have also presented complex-valued or lorentzian solutions that do not lead to contact structures on Bianchi type V and specialized forms of Bianchi type VI and VII spaces.

Then we have discussed a family of complex solutions of the euclidean topologically massive abelian gauge theory on \mathcal{R}^3 in cartesian coordinates. These complex (anti-)self-dual solutions are determined by (anti-)holomorphic functions. Their real parts accordingly yield real contact structures. The orthogonality relation of these contact structures separates the solution set into self-dual and anti-self-dual classes. Thus we call the respective contact structures self-dual and anti-self-dual.

We have also applied the curl transformation to the euclidean topologically massive abelian gauge theory on \mathcal{R}^3 . An arbitrary field or potential are given in terms of a vector tangent to a sphere whose radius is determined by the topological mass in the transform space. Meanwhile a gauge transformation corresponds to a vector normal to this sphere in this space. The spherical curl transformation is known to be equivalent to the Radon transformation.

Then we have discussed the sense of quantization of the topological mass: $\nu = ng^2$ on an example. Our discussion suggests the existence of a fundamental length scale $L = 2\pi/g^2 = 2\pi n/\nu$ in topologically massive gauge theories. The strength of the gauge potential is given by the gauge coupling constant $\nu/g = ng$. The fundamental length scale is the least common multiple of intervals over which the gauge potential and the transformation are single-valued and periodic for any integer n in addition to the fact that they have a smaller period $l = L/n$. A similar reasoning has already been used for the abc-flow in fluid dynamics. In the topologically massive gauge theories this length scale is naturally determined by the gauge coupling constant g .

The approaches here are mostly motivated by fluid dynamics and plasma physics or magneto-hydrodynamics. To the knowledge of the author, the connection of these structures with the topologically massive gauge theories has been overlooked in the literature. Moreover, we have not yet discussed other important solutions. The three approaches here have relative advantages over each other from different points of view. Yet, they together point

out richer and more interesting structures underlying topologically massive theories including gravity. These can contribute not only to the topologically massive theories but to our basic concepts such as space, time and field as well.

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A Classification of Trkalian Fields

We reproduce the following proposition from [13] in terms of 1-forms.

Proposition. Every 1-form α can be written as the real part of the complex 1-form

$$a = e^{ig} df,$$

$\alpha = \text{Re}\{a\}$, where $g : R^3 \longrightarrow R$ and $f : R^3 \longrightarrow C$.

Proof. It is always possible to construct the Monge potentials g, h, k so that a 1-form α can be expressed in Clebsch form

$$\alpha = kdg + dh.$$

If we define

$$f = (h + ik)e^{-ig}, \quad f^* = (h - ik)e^{ig},$$

we find

$$k = -\frac{i}{2} (fe^{ig} - f^*e^{-ig}), \quad h = \frac{1}{2} (fe^{ig} + f^*e^{-ig}).$$

Then we can write α as

$$\begin{aligned}\alpha &= \frac{1}{2} (e^{ig} df + e^{-ig} df^*) \\ &= \text{Re}\{a\}.\end{aligned}$$

The authors of [13] impose the constraint that a is Beltrami (Trkalian): $*da = a$ ($\nu = 1$) on the $y^1 = g$, $y^2 = f$, $y^3 = f^*$ system which is a diffeomorphism of flat \mathcal{R}^3 with coordinates x^1, x^2, x^3 . They identify precisely two classes of solutions: in the cartesian and the spherical coordinates. In the cartesian coordinates they conclude

$$ds^2 = dg^2 + dw dw^*,$$

where $g = z$ and f is a holomorphic function of $w = x + iy$. We consider the cartesian case including the topological mass.

B Curl Transformation

The vectors $\vec{Q}_\lambda(\vec{k})$ which form a complex basis in Fourier space were introduced in [14]. These possess the following properties

$$\begin{aligned}\vec{Q}_\lambda^*(\vec{k}) \cdot \vec{Q}_\mu(\vec{k}) &= \delta_{\lambda\mu}, & \lambda, \mu &= 0, \pm 1, \\ \sum_\lambda \vec{Q}_{\lambda i}^*(\vec{k}) \vec{Q}_{\lambda j}(\vec{k}) &= \delta_{ij}, & i, j &= 1, 2, 3.\end{aligned}$$

Thus the eigen-functions $\vec{\chi}_\lambda(\vec{x}|\vec{k})$ form an orthogonal and complete set

$$\begin{aligned}\int \vec{\chi}_\lambda^*(\vec{x}|\vec{k}) \cdot \vec{\chi}_\mu(\vec{x}|\vec{k}') d^3x &= \delta_{\lambda\mu} \delta(\vec{k} - \vec{k}'), \\ \sum_\lambda \int \vec{\chi}_{\lambda i}^*(\vec{x}|\vec{k}) \vec{\chi}_{\lambda j}(\vec{x}'|\vec{k}) d^3k &= \delta_{ij} \delta(\vec{x} - \vec{x}'),\end{aligned}$$

[14]. The vectors $\vec{Q}_\lambda(\vec{k})$ satisfy the following relations

$$\begin{aligned}
\vec{k} \times \vec{Q}_\lambda(\vec{k}) &= -i\lambda k \vec{Q}_\lambda(\vec{k}), & \lambda = 0, \pm 1, \\
\vec{k} \cdot \vec{Q}_\lambda(\vec{k}) &= 0, & \lambda = \pm 1, \\
\vec{Q}_\lambda^*(\vec{k}) &= -\vec{Q}_{-\lambda}(\vec{k}), \\
\vec{Q}_\lambda(\vec{k}) &= \vec{Q}_\lambda(\vec{\kappa}),
\end{aligned}$$

which are useful in simplifying the expressions.

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